ON THE COMPLEMENT OF THE PROJECTIVE HULL IN $\mathbf{C}^n$.

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ABSTRACT

We prove that if K is a compact subset of an affine variety $\Omega = \mathbf{P}^n - D$ (where D is a projective hypersurface) and if K is contained in a closed analytic subvariety $V \subset \Omega$, then the projective hull $\widehat{K}$ has the property that $\widehat{K} \cap \Omega \subset V$. If V is smooth and 1-dimensional, then $\widehat{K} \cap \Omega$ is also closed in Ω. The result has applications to graphs in $\mathbf{C}^2$ of functions in the disk algebra.

Let X be a compact set in $\mathbf{C}^n$. In [1], R. Harvey and the first author introduce a generalization of the polynomial hull of X, which they call the *projective hull of X in* $\mathbf{C}^n$, and denote by $\widehat{X}$. For $d = 1, 2, ...$ let $\mathcal{P}_d$ denote the space of all polynomials on $\mathbf{C}^n$ of degree $\leq d$. Then by definition a point z lies in $\widehat{X}$ if there exists a constant $C \geq 1$ such that for all d and for all $P \in \mathcal{P}_d$ with $\sup_X |P| \leq 1$, we have

$$|P(z)| \ \leq \ C^d.$$

Fix a point $a = (a_1, ..., a_n) \in \mathbf{C}^n - X$. For $d = 1, 2, ...$ set

$$\lambda_d(a) \ \equiv \ \sup\{|P(a)| : P \in \mathcal{P}_d \text{ with } \|P\|_X \leq 1\}$$

Note that a lies in the complement of $\widehat{X}$ if and only if for each $M \geq 1$ there exist arbitrarily large d with $\lambda_d(a) \geq M^d$.

We call a set X *non-algebraic* if every polynomial vanishing on X is identically zero. We restrict our attention to such non-algebraic sets. We study the function λ_d and, in Theorem 2 below, give a result on the complement of $\widehat{X}$ in certain cases, generalizing Theorem 9.2 in [1].

Theorem 1. *Let X be as above. Fix $M \geq 1$. Fix d and let $\lambda_d = \lambda_d(0)$. Then $\lambda_d \geq M^d$ if and only if there exist polynomials $A_1, ..., A_n$ on $\mathbf{C}^n$ such that $\zeta_1 A_1(\zeta) + \cdots + \zeta_n A_n(\zeta) \in \mathcal{P}_d$ with $|1 - \sum_k \zeta_k A_k| \leq \frac{1}{M^d}$ on X.*

We use the following Banach distance formula.

Lemma 1. *Let W be a normed linear space, dim$(\mathcal{W}) = N$, and let $\mathcal{V}$ be a subspace of $\mathcal{W}$ with dim$(\mathcal{V}) = N - 1$. Fix $x \in \mathcal{W} - \mathcal{V}$. Let L be the linear functional on $\mathcal{W}$ with $L = 0$ on $\mathcal{V}$ and $L(x) = 1$. Let $\delta = \text{dist}(x, \mathcal{V})$ Then*

$$\|L\| \ = \ \frac{1}{\delta}.$$

*Partially supported by the N.S.F.

NOTE . See the Appendix for a proof.

Proof of Theorem 1. Let $\mathcal{W} = \mathcal{P}_d$. For $P \in \mathcal{P}_d$ put $\|P\| = \sup_X |P|$. Note that $\|P\| = 0$ only when $P = 0$ by the hypothesis that X is non-algebraic.

A basis for $\mathcal{W}$ is the set of monomials $\zeta^\alpha = \zeta_1^{\alpha_1}\zeta_2^{\alpha_2}\cdots\zeta_n^{\alpha_n}$ with $|\alpha| = \alpha_1+\cdots+\alpha_n \leq d$. Let $\mathcal{V}$ be the subspace of $\mathcal{W}$ spanned by $\{\zeta^\alpha : |\alpha| > 0\}$. Let x denote the constant polynomial 1. Define L to be the linear functional $L : \mathcal{W} \to \mathbf{C}$ given by $L(P) = P(0)$. Then $\lambda_d = \|L\|$ in the dual space of $\mathcal{W}$. Put $\delta =$ the distance from 1 to $\mathcal{V}$, so

$$\delta \;=\; \inf\left\|1 - \sum_{0<|\alpha|\leq d} c_\alpha \zeta^\alpha\right\|.$$

By Lemma 1, $\lambda_d \geq M^d$ if and only if $\|L\| \geq M^d$ if and only if $\frac{1}{\delta} \geq M^d$ if and only if $\delta \leq \frac{1}{M^d}$ if and only if

$$\left|1 - \sum_{k=1}^{n} \zeta_k A_k\right| \;\leq\; \frac{1}{M^d} \quad \text{on} \;\; X$$

for polynomials $A_1, ..., A_n \in \mathcal{P}_{d-1}$. We are done. ■

THEOREM 2. *Let Σ be a closed complex analytic subvariety of $\mathbf{C}^n$. Let K be a compact set contained in Σ. Then $\widehat{K} \subset \Sigma$.*

NOTE 1. In the case that $\Sigma \subset \mathbf{C}^2$ is the graph of an entire function f on $\mathbf{C}$, with f not a polynomial, the result that $K \subset \Sigma$ implies $\widehat{K} \subset \Sigma$ is proved in Theorem 9.2 in [1].

Proof of Theorem 2. Fix $a = (a_1, ..., a_n) \in \mathbf{C}^n - \Sigma$. By definition $\lambda_d(a) = \sup |P(a)|$ taken over all $P \in \mathcal{P}_d$ with $\|P\|_K \leq 1$. We make the following

Claim: Fix $M \geq 1$. Then $\lambda_d(a) > M^d$ for all large d.

Proof of Claim:

Case 1: $a = 0$ so $\lambda_d(a) = \lambda_d$. Then $0 \in \mathbf{C}^n - \Sigma$. It is known that we may choose a function H_1, holomorphic on $\mathbf{C}^n$ with $H_1 = 0$ on Σ and $H_1(0) = 1$. (See, for example, [2 Ch. VIII.A, Thm. 18].) Put $H = 1 - H_1$. Then H is holomorphic on $\mathbf{C}^n$ with $H = 1$ on Σ and $H(0) = 0$. Now H has a power series expansion

$$H(\zeta) \;=\; \sum_\alpha c_\alpha \zeta^\alpha$$

convergent on $\mathbf{C}^n$. Fix $R > 0$. There exists a constant C_R such that

$$|c_\alpha| \;\leq\; \frac{C_R}{R^{|\alpha|}} \qquad \text{for all} \;\; \alpha$$

Fix d. Put $P_d(\zeta) = \sum_{|\alpha|\leq d} c_\alpha \zeta^\alpha$ and put $\mathcal{E}_d(\zeta) = \sum_{|\alpha|>d} c_\alpha \zeta^\alpha$. Note that $P_d(0) = 0$. Then $H = P_d + \mathcal{E}_d$ on $\mathbf{C}^n$. In particular, $1 = P_d + \mathcal{E}_d$ on K, so

$$|1 - P_d| \;\leq\; |\mathcal{E}_d| \quad \text{on} \;\; K. \tag{1}$$

Fix R_0 so that $K \subset \{\zeta : |\zeta_k| \leq R_0 \text{ for } 1 \leq k \leq n\}$. Choose $\zeta \in K$. Then

$$|\mathcal{E}_d(\zeta)| \;\leq\; \sum_{|\alpha|>d} |c_\alpha||\zeta^\alpha| \;\leq\; \sum_{|\alpha|>d} |c_\alpha| R_0^{|\alpha|}.$$

Fix R such that $\frac{R_0}{R} \leq \frac{1}{2M}$. Since $|c_\alpha| \leq \frac{C_R}{R^{|\alpha|}}$, we have

$$\sum_{|\alpha|>d} |c_\alpha| R_0^{|\alpha|} \;\leq\; C_R \sum_{|\alpha|>d} \left(\frac{R_0}{R}\right)^{|\alpha|}$$

Put $t = \frac{R_0}{R}$. Then

$$\sum_{|\alpha|>d} t^{|\alpha|} \;=\; \sum_{k=d+1}^{\infty} \sum_{|\alpha|=k} t^{|\alpha|} \;=\; \sum_{k=d+1}^{\infty} \binom{n+k-1}{n-1} t^k \;\leq\; \frac{1}{M^d} \quad \text{for large } d.$$

Hence,

$$|\mathcal{E}_d(\zeta)| \;\leq\; \frac{C_R}{M^d} \;\leq\; \frac{1}{(M/2)^d} \quad \text{for large } d. \tag{2}$$

From (1) and Theorem 1 we now get

$$\lambda_d \;\geq\; \left(\frac{M}{2}\right)^d \quad \text{for large } d.$$

Since M was arbitrary, the claim is proved Case 1.

Case 2: $a \in \mathbf{C}^n - \Sigma$**.** Let $\chi : \mathbf{C}^n \to \mathbf{C}^n$ be the translation $\chi(\zeta) = \zeta - a$. Then $\chi(a) = 0$ and χ takes the space $\mathcal{P}_d$ isomorphically onto itself. Put $K' = \chi(K)$ and $\Sigma' = \chi(\Sigma)$. Since $a \notin \Sigma$, $0 \notin \Sigma'$ and so by Case 1 there exists $P' \in \mathcal{P}_d$ with $\|P'\|_{K'} \leq 1$, $|1 - P'| \leq \frac{1}{M^d}$ on K' and $P'(0) = 0$.

By Case 1, $\lambda_d \geq M^d$ for large d, so there exists a polynomial Q' in $\mathcal{P}_d$ with $\|Q'\|_{K'} \leq 1$ and $|Q'(0)| \geq M^d$. Now put $Q = Q' \circ \chi$. Then $Q \in \mathcal{P}_d$, $|Q(a)| \geq M^d$ and $\|Q\|_K \leq 1$. so $\lambda_d(a) \geq M^d$. Case 2 is done and so the claim holds.

If $a \in \mathbf{C}^n - \Sigma$, then a does not belong to $\widehat{K}$. So $\widehat{K} \subset \Sigma$. We are done. ∎

Corollary 1. *Let V be a closed complex submanifold of dimension one in $\mathbf{C}^n$ and $K \subset V$ a compact subset. Then $\widehat{K}$ is a closed subset of $\mathbf{C}^n$.*

Note . It is not true in general that K compact in $\mathbf{C}^n$ implies that $\widehat{K}$ is closed in $\mathbf{C}^n$.

Proof of Corollary 1. In the notation of [1], if K is a compact set in $\mathbf{C}^n$, $\mathcal{S}_K$ is the family of functions $\varphi = \frac{1}{d}\log|P|$, $P \in \mathcal{P}_d$, $|P| \leq 1$ on K. We define the extremal function

$$\Lambda_K(x) \;=\; \sup_{\varphi \in \mathcal{S}_K} \varphi(x).$$

Then for $x \in \mathbf{C}^n$, $\Lambda_K(x) < \infty \iff x \in \widehat{K}$. That is, $\widehat{K} \cap \mathbf{C}^n = \{x \in \mathbf{C}^n : \Lambda_K(x) < \infty\}$. Since $\Lambda_K \equiv \infty$ in $\mathbf{C}^n - V$, we conclude from Theorem 7.3 in [1] that in every connected component of V_K either $\Lambda_K \equiv \infty$ or Λ_K is a locally bounded harmonic function.

Suppose now that $\{x_\nu\}$ is a sequence of points in $\widehat{K}$ converging to a point x. Then $x \in V$. We must show that $x \in \widehat{K}$.

We may assume that $x \notin K$, so x lies in some connected component Ω of $V - K$ For large ν, $x_\nu \in \Omega$. Since $\Lambda_K(x_\nu) < \infty$, we have by Theorem 7.3 of [1] quoted above, that Λ_K is locally bounded on Ω. In particular, $L_K(x)$ is finite, and so $x \in \widehat{K}$. ∎

Note 2. The projective hull, as introduced in [1], is a subset $\widehat{K} \subset \mathbf{P}^n$ associated to any compact set $K \subset \mathbf{P}^n$. In an affine chart $\mathbf{C}^n = \mathbf{P}^n - \mathbf{P}^{n-1}$, the set $\widehat{K} \cap \mathbf{C}^n$ agrees with the definition given above. In what follows $\widehat{K}$ will refer to this full projective hull.

Corollary 2. *Let $D \subset \mathbf{P}^n$ be a complex hypersurface in complex projective space, and suppose $\Sigma \subset \mathbf{P}^n - D$ is a closed subvariety of the complement. If K is a compact subset of Σ, then $\widehat{K} \cap (\mathbf{P}^n - D) \subset \Sigma$. Moreover, if Σ is smooth of dimension one, then $\widehat{K} \cap (\mathbf{P}^n - D)$ is closed in $\mathbf{P}^n - D$.*

Proof. The divisor D can be realized as a hyperplane section $D = v(\mathbf{P}^n) \cap \mathbf{P}^{N-1}$ under a Veronese embedding $v : \mathbf{P}^n \to \mathbf{P}^N$. By Proposition 3.2 in [1] we know that $\widehat{v(K)} = v(\widehat{K})$. We now apply Theorem 2 to $v(K) \subset v(\Sigma) \subset \mathbf{C}^N = \mathbf{P}^N - \mathbf{P}^{N-1}$, and the first statement follows. If Σ is a smooth curve, then $\widehat{v(K)}$ is closed in $\mathbf{C}^N$ by Corollary 1. Hence, $v^{-1}(\widehat{v(K)}) = v^{-1}(v(\widehat{K})) = \widehat{K}$ is closed in $\mathbf{P}^n - D$. ∎

Corollary 3. *Let $K \subset \Sigma \subset \mathbf{P}^n - D$ be as in Corollary 2. Suppose $\widehat{K} \cap (\mathbf{P}^n - D) \subset\subset \mathbf{P}^n - D$. Then $\widehat{K} \cap D = \emptyset$.*

Proof. Suppose there exists a point $x \in \widehat{K} \cap D$. Then by Theorem 11.1 in [1], there exists a positive current T of bidimension (1,1) with

$$\mathrm{supp} T \ \subset \ \widehat{K}^- \qquad \text{and} \qquad dd^c T \ = \ \mu - \delta_x \tag{3}$$

where μ is a probability measure on K and $\widehat{K}^-$ denotes the closure of K. Let $U, V \subset \mathbf{P}^n$ be disjoint open subsets with $\widehat{K} \cap (\mathbf{P}^n - D) \subset U$ and $D \subset V$. Then by our hypothesis and the first part of (3), we have $T = \chi_U T + \chi_V T$. We conclude from the second part of (3) that $dd^c(\chi_U T) = \mu$ and $dd^c(\chi_V T) = -\delta_x$, both of which are impossible since $(dS, 1) = 0$ for any current S of dimension $2n - 1$. ∎

Corollary 4. *Suppose $K = \gamma$ is a disjoint union of real closed curves contained in a smooth complex analytic (non-algebraic) curve $\Sigma \subset \mathbf{P}^n - D$ with D as above. Suppose that $\Sigma - \gamma$ has only one unbounded component. Then $\widehat{\gamma} - \gamma$ is exactly the union of the bounded components of $\Sigma - \gamma$.*

Proof. By Corollary 2, $\widehat{\gamma} \cap (\mathbf{P}^n - D) \subset \Sigma$. As seen in the proof of Corollary 1, if one point $x \in \Sigma - \gamma$ belongs to $\widehat{\gamma}$, then the entire connected component of x in $\Sigma - \gamma$ belongs to $\widehat{\gamma}$. By [1, Cor. 3.4] every bounded component of $\Sigma - \gamma$ lies in $\widehat{\gamma}$. By Sadullaev's Theorem (cf. [1, §7]), if the unbounded component of $\Sigma - \gamma$ lies in $\widehat{\gamma}$, then Σ is algebraic. Thus $\widehat{\gamma} \cap (\mathbf{P}^n - D)$ is exactly the union of the bounded components of $\Sigma - \gamma$. We now apply Corollary 3. ∎

Note 3. We can find applications of Corollary 4 by considering graphs of certain functions in the disk algebra.

Let Γ be the unit circle in the ζ-plane and let $A(\Gamma) = \{\varphi\big|_\Gamma : \varphi$ is analytic in $|\zeta| < 1$ and continuous in $|\zeta| \leq 1\}$. Let K denote the graph of φ over Γ, so

$$K \;=\; \{(\zeta, \varphi(\zeta)) : \zeta \in \Gamma\} \;\subset\; \mathbf{C}^2.$$

In the following examples the projective hull of K in $\mathbf{P}^2$ is $\widehat{K} = \{(\zeta, \varphi(\zeta)) : |\zeta| \leq 1\}$.

(1) Let $\varphi(\zeta) = \log(\zeta - 2)$. Then $K = \{(\zeta, \log(\zeta - 2)) : \zeta \in \Gamma\}$, so K is contained in the closed complex submanifold $\Sigma = \{(\zeta, w) : e^w = \zeta - 2\}$ of $\mathbf{C}^2$.

(2) Let $\varphi(\zeta) = \sqrt{e^{2\zeta} + 5\zeta + 20}$ and $\Sigma = \{(\zeta, w) : w^2 = e^{2\zeta} + 5\zeta + 20\}$. The graph K of φ over Γ is contained in Σ

(3) Let $\varphi(\zeta)$ be the restriction to Γ of a meromorphic function F defined on all of $\mathbf{C}$ whose poles lies outside the unit disk. Write $F = E_1/E_2$ where E_1 and E_2 are entire functions without common zeros. Then $\Sigma = \{(\zeta, w) : E_2(\zeta)w = E_1(\zeta)\}$ (the graph of F) is a closed submanifold of $\mathbf{C}^2$ which contains K

(4) Let

$$\varphi(\zeta) \;=\; \sum_{\nu=1}^{\infty} \frac{c_\nu}{\zeta - a_\nu}$$

where $\sum_\nu |c_\nu| < \infty$, each $|a_\nu| > 1$, and $\lim_{\nu\to\infty} a_\nu = a$ with $|a| > 1$. Then K is contained in $\Sigma = \{(\zeta, \varphi(\zeta)) : \zeta \neq a$ or a_ν for any $\nu\}$, which is a closed submanifold of $\mathbf{C}^2 - \{\zeta = a\} = \mathbf{P}^2 - \mathbf{P}^1_\infty - \{\zeta = a\}$.

Appendix: Proof of the Banach Distance Formula.

Fix $f \in \mathcal{V}$. Then $L(x - f) = L(x) = 1$. Hence, $1 = |L(x - f)| \leq \|L\| \cdot \|x - f\|$. Hence, $1 \leq \|L\| \cdot \delta$, so $\|L\| \geq \frac{1}{\delta}$.

Next, fix $g \in \mathcal{W}$, $g = f + tx$ for $t \in \mathbf{C} - \{0\}$, $f \in \mathcal{V}$. We have $L(g) = t$. Now $g = t(x + \frac{f}{t})$, so $\|g\| = |t|\,\|x + \frac{f}{t}\| \geq |t| \cdot \delta$, since $-\frac{f}{t} \in \mathcal{V}$. So $|L(g)| = |t| \leq \frac{1}{\delta}\|g\|$. Thus, $\|L\| \leq \frac{1}{\delta}$, and therefore $\|L\| = \frac{1}{\delta}$. ∎

References.

[1] F. R. Harvey and H. B. Lawson, Jr. *Projective hulls and the projective Gelfand transform*, Asian J. Math. **10** (2006), 607-646.

[2] R. C. Gunning and H. Rossi, Analytic Functions of Several Complex Variables, Prentice-Hall, Englewood Cliffs, N. J., 1965.